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Matematika III

DIFFERENSIAL VEKTOR
(Turunan Parsial Fungsi Vektor)

TURUNAN PARSIAL

- Turunan parsial untuk fungsi vektor dua variabel atau lebih, prinsipnya sama dengan definisi turunan fungsi vektor satu variabel, dimana semua variabel dianggap konstan, kecuali satu, yaitu variabel terhadap apa fungsi vektor itu diturunkan.

TURUNAN PARSIAL FUNGSI VEKTOR

Misalkan $\mathbf{A}$ adalah sebuah fungsi vektor yang tergantung kepada variabel skalar x, y, dan z, maka kita tuliskan $\mathbf{A} = \mathbf{A}(x, y, z)$. Ketiga turunan parsialnya didefinisikan sebagai berikut:

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x + \Delta x, y, z) - \mathbf{A}(x, y, z)}{\Delta x}$$

$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y + \Delta y, z) - \mathbf{A}(x, y, z)}{\Delta y}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x, y, z + \Delta z) - \mathbf{A}(x, y, z)}{\Delta z}$$

adalah masing-masing turunan parsial dari $\mathbf{A}$ terhadap x, y, dan z jika limitnya ada.

TURUNAN PARSIAL FUNGSI VEKTOR

Jika fungsi vektor $\mathbf{A}(x, y, z) = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}$ dengan fungsi skalar-fungsi skalar $A_1(x, y, z)$, $A_2(x, y, z)$, dan $A_3(x, y, z)$ mempunyai turunan parsial terhadap variabel x, y, dan z, maka $\mathbf{A}(x, y, z)$ juga mempunyai turunan variabel terhadap x, y, dan z yang dirumuskan sebagai berikut:

$$\frac{\partial \mathbf{A}}{\partial x} = \frac{\partial A_1}{\partial x}\mathbf{i} + \frac{\partial A_2}{\partial x}\mathbf{j} + \frac{\partial A_3}{\partial x}\mathbf{k}$$

$$\frac{\partial \mathbf{A}}{\partial y} = \frac{\partial A_1}{\partial y}\mathbf{i} + \frac{\partial A_2}{\partial y}\mathbf{j} + \frac{\partial A_3}{\partial y}\mathbf{k}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \frac{\partial A_1}{\partial z}\mathbf{i} + \frac{\partial A_2}{\partial z}\mathbf{j} + \frac{\partial A_3}{\partial z}\mathbf{k}$$

SIFAT TURUNAN PARSIAL

Misalkan **A** dan **B** adalah fungsi-fungsi vektor dan ϕ adalah fungsi skalar x, y, dan z dan dapat dideferensialkan terhadap ketiga variabel tersebut, maka berlaku

i. $\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B}) = \frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}$

ii. $\frac{\partial}{\partial x}(\phi \mathbf{A}) = \phi\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \phi}{\partial x}\mathbf{A}$

iii. $\frac{\partial}{\partial x}(\mathbf{A}.\mathbf{B}) = \mathbf{A}.\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}.\mathbf{B}$

iv. $\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\times\mathbf{B}$

v. $\frac{\partial^2}{\partial y \partial x}(\mathbf{A}.\mathbf{B}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A}.\mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A}.\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}.\mathbf{B}\right\}$

$$= \mathbf{A}.\frac{\partial^2 \mathbf{B}}{\partial y \partial x} + \frac{\partial \mathbf{A}}{\partial y}.\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}.\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y \partial x}.\mathbf{B}$$

Pembuktian Sifat Turunan Parsial

i. Berdasarkan definisi 3.3, maka

$$\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})$$

$$= \lim_{\Delta x\to 0}\frac{[\mathbf{A}(x+\Delta x,y,z)+\mathbf{B}(x+\Delta x,y,z)]-[\mathbf{A}(x,y,z)+\mathbf{B}(x,y,z)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\mathbf{A}(x+\Delta x,y,z)+\mathbf{B}(x+\Delta x,y,z)-\mathbf{A}(x,y,z)-\mathbf{B}(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{[\mathbf{A}(x+\Delta x,y,z)-\mathbf{A}(x,y,z)]+[\mathbf{B}(x+\Delta x,y,z)-\mathbf{B}(x,y,z)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\mathbf{A}(x+\Delta x,y,z)-\mathbf{A}(x,y,z)}{\Delta x} + \lim_{\Delta x\to 0}\frac{\mathbf{B}(x+\Delta x,y,z)-\mathbf{B}(x,y,z)}{\Delta x}$$

Sehingga

$$\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B}) = \frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}$$

ATURAN RANTAI

Misalkan $\mathbf{F} = \mathbf{F}(x, y, z)$ adalah fungsi vektor yang dapat dideferensialkan terhadap variabel x, y, dan z, dimana $x = x(s, t, u)$, $y = y(s, t, u)$, dan $z = z(s, t, u)$ adalah fungsi-fungsi skalar yang dapat dideferensialkan terhadap variabel s, t, dan u, maka bentuk fungsi tersusun $\mathbf{F}$ dapat dituliskan dengan

$$\mathbf{F} = \mathbf{F}[x(s,t,u), y(s,t,u), z(s,t,u)]$$

Turunan parsial $\mathbf{F}$ terhadap variabel s, t, dan u dapat diberikan sebagai berikut:

$$\frac{\partial \mathbf{F}}{\partial s} = \frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial \mathbf{F}}{\partial u} = \frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial u}$$

SOAL#1

Jika $\mathbf{F} = xyz^2\mathbf{i} + yz^2\mathbf{j} + 2xy^2\mathbf{k}$, tentukanlah (a)$\frac{\partial \mathbf{F}}{\partial x}$, (b)$\frac{\partial \mathbf{F}}{\partial y}$, (c)$\frac{\partial \mathbf{F}}{\partial z}$

Penyelesaian

(a) $\frac{\partial \mathbf{F}}{\partial x} = \frac{\partial}{\partial x}(xyz^2\mathbf{i} + yz^2\mathbf{j} + 2xy^2\mathbf{k})$
$= yz^2\mathbf{i} + 0 + 2y^2\mathbf{k}$
$= yz^2\mathbf{i} + 2y^2\mathbf{k}$

(b) $\frac{\partial \mathbf{F}}{\partial y} = \frac{\partial}{\partial y}(xyz^2\mathbf{i} + yz^2\mathbf{j} + 2xy^2\mathbf{k})$
$= xz^2\mathbf{i} + z^2\mathbf{j} + 4xy\mathbf{k}$

(c) $\frac{\partial \mathbf{F}}{\partial z} = \frac{\partial}{\partial z}(xyz^2\mathbf{i} + yz^2\mathbf{j} + 2xy^2\mathbf{k})$
$= 2xyz\mathbf{i} + 2yz\mathbf{j} + 0$
$= 2xyz\mathbf{i} + 2yz\mathbf{j}$

SOAL#2

Misalkan $\mathbf{A} = e^{\sin x^2yz}\mathbf{i} + \cos x^3y^2z\mathbf{j} + \ln x^4y^3z\,\mathbf{k}$. Tentukan(a)$\frac{\partial \mathbf{A}}{\partial x}$,(b)$\frac{\partial \mathbf{A}}{\partial y}$,(c)$\frac{\partial \mathbf{A}}{\partial z}$

Penyelesaian

(a) $\frac{\partial \mathbf{A}}{\partial x} = \frac{\partial}{\partial x}\left(e^{\sin x^2yz}\mathbf{i} + \cos x^3y^2z\mathbf{j} + \ln x^4y^3z\,\mathbf{k}\right)$

$= 2xyz\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 3x^2y^2z\sin x^3y^2z\mathbf{j} + \frac{4x^3y^3z}{x^4y^3z}\mathbf{k}$

$= 2xyz\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 3x^2y^2z\sin x^3y^2z\mathbf{j} + \frac{4}{x}\mathbf{k}$

(b) $\frac{\partial \mathbf{A}}{\partial y} = \frac{\partial}{\partial y}\left(e^{\sin x^2yz}\mathbf{i} + \cos x^3y^2z\mathbf{j} + \ln x^4y^3z\,\mathbf{k}\right)$

$= x^2z\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 2x^3yz\sin x^3y^2z\mathbf{j} + \frac{3x^4y^2z}{x^4y^3z}\mathbf{k}$

$= x^2z\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 2x^3yz\sin x^3y^2z\mathbf{j} + \frac{3}{y}\mathbf{k}$

(c) $\frac{\partial \mathbf{A}}{\partial z} = \frac{\partial}{\partial z}\left(e^{\sin x^2yz}\mathbf{i} + \cos x^3y^2z\mathbf{j} + \ln x^4y^3z\,\mathbf{k}\right)$

$= x^2y\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 2x^3y^2\sin x^3y^2z\mathbf{j} + \frac{x^4y^3}{x^4y^3z}\mathbf{k}$

$= x^2y\cos x^2yz\, e^{\sin x^2yz}\mathbf{i} + 2x^3y^2\sin x^3y^2z\mathbf{j} + \frac{1}{z}\mathbf{k}$

LATIHAN (PR)

1 Jika $\mathbf{Z} = 3x^2\mathbf{i} - y^2\mathbf{j}$, dengan $x = 2s + 7t$ dan $y = 5st$, tentukan $\frac{\partial z}{\partial t}$ dan nyatakan dalaam bentuk s dan t.

2 Jika $\mathbf{F} = \sin xy^2z\,\mathbf{i} + 2yz\mathbf{j} + z^2\mathbf{k}$. Tentukan $\frac{\partial^2 \mathbf{F}}{\partial x^2}$

3 Jika $\mathbf{W} = (x^2 + y^2)\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$, $x = st, y = s - t, z = s + 2t$, tentukan $\frac{\partial \mathbf{W}}{\partial t}$

SOAL#3

Jika $\mathbf{Z} = 3x^2\mathbf{i} - y^2\mathbf{j}$, dengan $x = 2s + 7t$ dan $y = 5st$, tentukan $\frac{\partial z}{\partial t}$ dan nyatakan dalaam bentuk s dan t.

SOAL#4

Jika $\mathbf{F} = \sin xy^2 z\,\mathbf{i} + 2yz\mathbf{j} + z^2\mathbf{k}$. Tentukan $\frac{\partial^2 \mathbf{F}}{\partial x^2}$

Penyelesaian

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin xy^2 z\,\mathbf{i} + 2yz\mathbf{j} + z^2\mathbf{k})\right]$$

$$= \frac{\partial}{\partial x}(y^2 z \;\ldots\ldots\ldots\ldots + \;\ldots\; + \;\ldots\;)$$

$$= \;\ldots\ldots\ldots\ldots\ldots$$

SOAL#5

Jika $\mathbf{W} = (x^2 + y^2)\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}, x = st, y = s - t, z = s + 2t$, tentukan $\frac{\partial \mathbf{W}}{\partial t}$

Penyelesaian

$$\frac{\partial \mathbf{W}}{\partial t} = \frac{\partial \mathbf{W}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{W}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{W}}{\partial z}\frac{\partial z}{\partial t}$$

$$= (\ldots + \ldots)(s) + (2y\mathbf{i} + \ldots)(-1) + (\ldots)(\ldots)$$

$$= [2(\ldots)\mathbf{i} + (\ldots - \ldots)\mathbf{k}](\ldots) + [\ldots(\ldots - \ldots)\mathbf{i} - (\ldots)\mathbf{k}](-1)$$

$$+ \ldots(\ldots + \ldots)\mathbf{j}(\ldots)$$

$$= \ldots\mathbf{i} + (\ldots - \ldots)\mathbf{k} - (\ldots - \ldots)\mathbf{i} - st\mathbf{k} + (\ldots + \ldots)\mathbf{j}$$

$$= (\ldots\ldots - \ldots + \ldots)\mathbf{i} + (\ldots + \ldots)\mathbf{j} + (s^2 - 3st)\mathbf{k}$$

LATIHAN#1

- Jika $\mathbf{A} = \cos xy\,\mathbf{i} + (3xy - 2x^2)\mathbf{j} - (3x + 2y)\mathbf{k}$,
- Carilah : $\frac{\partial \mathbf{A}}{\partial x}, \frac{\partial \mathbf{A}}{\partial y}, \frac{\partial^2 \mathbf{A}}{\partial x^2}, \frac{\partial^2 \mathbf{A}}{\partial y^2}, \frac{\partial^2 \mathbf{A}}{\partial x \partial y}, \frac{\partial^2 \mathbf{A}}{\partial y \partial x}$

LATIHAN#2

Jika $\mathbf{A} = x^2yz\mathbf{i} - 2xz^3\mathbf{j} + xz^2\mathbf{k}$ dan $\mathbf{B} = 2z\mathbf{i} + y\mathbf{j} + x^2\mathbf{k}$, carilah $\frac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B})$ di titik (1,0,-2)

LATIHAN#3

Misalkan $\mathbf{F} = e^{xyz}\mathbf{i}$, dimana $x = stu^2, y = st^2u + us, dan\ z = st + tu.$
Tentukan (a) $\frac{\partial \mathbf{F}}{\partial s}$, (b) $\frac{\partial \mathbf{F}}{\partial t}$, (c) $\frac{\partial \mathbf{F}}{\partial u}$

LATIHAN#4

Jika $\mathbf{F} = \sin xy^2 z\,\mathbf{i} + 2yz\mathbf{j} + z^2\mathbf{k}$, tentukanlah (a)$\frac{\partial \mathbf{F}}{\partial x}$, (b)$\frac{\partial \mathbf{F}}{\partial y}$, (c)$\frac{\partial \mathbf{F}}{\partial z}$